Chapter 6.

Applications of differentiation.

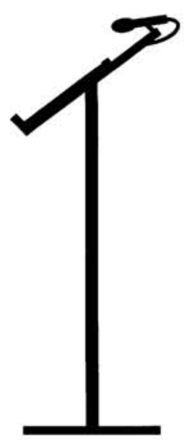
Situation

Certain medical personnel in a country are concerned about the rate at which a particular disease is spreading through the population. The disease was first properly identified in 2003 and by the beginning of 2005 the number of people in the country known to have been suffering from the disease was 2050. Figures for later years suggest that the number of people in the country known to be suffering from the disease approximately follows the rule

$$N = t^3 + 5t^2 + 10t,$$

where N is the number of sufferers t years after the disease was first introduced to the country.

One of the country's top doctors is planning a speech about the disease and asks you to supply answers to the six questions given below.
Assuming the above equation and the 2005 figures are correct obtain answers to the questions.



- Is it true that the figures suggest that the disease first entered the country around the beginning of 1994?
- How many people in the country suffered from the disease at the beginning of 2000, even though it had not been properly identified at that time?
- How many people in the country suffered from the disease at the beginning of 2010?
- What was the average number of <u>new sufferers per year</u> in this ten year period?
- What was the rate of increase in the number of sufferers (in new sufferers per year) at the beginning of 2010?
- If nothing is done to alter the spread of the disease what is the rate of increase in the number of sufferers likely to be (in new sufferers per year) by the beginning of
  - (a) the year 2040,
  - (b) the year 2050 ?

Rates of change.

Did you use differentiation in the situation on the previous page? The situation required a rate of change at a particular time to be determined. Differentiation gives a formula from which such a rate of change can be found.

In chapter 5 we used differentiation to find $\frac{dy}{dx}$, the rate of change of y with respect to x.

If other variables are involved, say volume, V, and time, t, then we can use differentiation to find $\frac{dV}{dt}$ the rate of change of volume with respect to time.

Example 1

If $P = 5t^2 + 6t$ find an expression for the rate of change of P with respect to t.

If $P = 5t^2 + 6t$ then $\frac{dP}{dt} = 10t + 6$.

The rate of change of P with respect to t is given by $10t + 6$.

Example 2

If $L = a^3 - 3a^2 + 5$ find the rate of change of L with respect to a when $a = 3$.

If $L = a^3 - 3a^2 + 5$ then $\frac{dL}{da} = 3a^2 - 6a$

If $a = 3$ then $\frac{dL}{da} = 9$

The rate of change of L with respect to a, when $a = 3$, is 9.

Alternatively the same answers can be obtained from a calculator.

$$\frac{d}{dt}(5t^2+6t) \qquad 10 \cdot t + 6$$

$$\frac{d}{da}(a^3-3a^2+5)\,|\,a=3 \qquad 9$$

Example 3

The volume of a sphere is increasing in such a way that the volume, V cm^3, at time t seconds is given by: $V = 7500 + 3600t - 150t^2$ for $0 \le t \le 12$.

Calculate (a) the volume when $t = 12$,

(b) an expression for the rate of change of volume with respect to time,

(c) the rate at which the volume is increasing (in cm^3/sec) when $t = 2$ and when $t = 10$.

(a) If $V = 7500 + 3600t - 150t^2$

then $V(12) = 7500 + 3600(12) - 150(12)^2$

$= 29\,100$

When $t = 12$ the volume is $29\,100$ cm^3.

(b) If $V = 7500 + 3600t - 150t^2$

then $\dfrac{dV}{dt} = 3600 - 300t$

The instantaneous rate of change in the volume with respect to time is given by

$3600 - 300t$.

(c) Using $\dfrac{dV}{dt} = 3600 - 300t$:

For $t = 2$, volume is increasing at 3000 cm^3/sec.

For $t = 10$, volume is increasing at 600 cm^3/sec.

Define $v(t)=7500+3600t-150t^2$ — Done

$v(12)$ — 29100

$\frac{d}{dt}(v(t))$ — $-300 \cdot t+3600$

$\frac{d}{dt}(v(t))|t=2$ — 3000

$\frac{d}{dt}(v(t))|t=10$ — 600

☛ As the above display suggests, answers can be obtained from a calculator without having to differentiate "by hand". Whilst you are encouraged to explore the capability of your calculator in this regard make sure you can use the appropriate calculus and algebraic methods yourself as well.

Exercise 6A

1. If $Q = 5r^2 + 3r - 4$ find an expression for the rate of change of Q with respect to r.

2. If $X = 3k + 3k^2 - 6k^3$ find an expression for the rate of change of X with respect to k.

3. If $T = 5r^3 - r^2 + 15r - 3$ find an expression for the rate of change of T with respect to r.

4. If $Q = 2p^4 + 3p^3 - 14p - 21$ find an expression for the rate of change of Q with respect to p.

5. If $P = (3t^2 - 2)(4t + 3)$ find an expression for the rate of change of P with respect to t.

6. If $A = 5t^2 + 6t - 1$ find the rate of change of A with respect to t when
   (a) $t = 1$, (b) $t = 2$, (c) $t = 3$.

7. If $P = 3a^2 + 4$ find the rate of change of P with respect to a when
   (a) $a = 2$, (b) $a = 3$, (c) $a = -4$.

8. If $A = \pi r^2$ find the rate of change of A with respect to r when
   (a) $r = 10$, (b) $r = 3$, (c) $r = \frac{70}{\pi}$.

9. If $A = 2\pi r^2 + 20\pi r$ find, in terms of π, the rate of change of A with respect to r when
   (a) $r = 3$, (b) $r = 7$, (c) $r = 10$.

10. If $V = \frac{4}{3}\pi r^3$ find, in terms of π, the rate of change of V with respect to r when
    (a) $r = 1$, (b) $r = 3$, (c) $r = 10$.

11. A goldfish breaks the water surface of a pond when collecting food and causes a circular ripple to emanate outwards. The radius of the circle, in metres, is given by $r = \frac{2t}{5}$ where t is the time in seconds after the goldfish caused the ripple to commence.
    (a) Find an expression for the area of the circle in terms of t.
    (b) Find the area of the circle after two seconds.
    (c) Find an expression for the rate at which the area is increasing with respect to t.
    (d) Find the instantaneous rate of increase of A when $t = 3$.

12. A colony of bacteria is increasing in such a way that the number of bacteria present after t hours is given by N where $N = 120 + 500t + 10t^3$.
    (a) Find the number of bacteria present initially (i.e. when $t = 0$).
    (b) Find the number of bacteria present when $t = 5$.
    (c) Find the average rate of increase, in bacteria/hour, in the first 5 hours.
    (d) Find an expression for the instantaneous rate of change of N with respect to time.
    (e) Find the rate the colony is increasing, in bacteria/hour, when
        (i) $t = 2$, (ii) $t = 5$, (iii) $t = 10$.

13. The total number of units, N, produced by a machinist, t hours into an 8 hour shift was found to approximately fit the mathematical model

$$N = 42t + 9t^2 - t^3 \quad \text{for } 0 \le t \le 8.$$

    (a) How many units did the machinist produce in the eight hours?
    (b) What was the machinist's average production rate, in units/hour, during the shift?
    (c) How many units did the machinist produce in the final hour?
    (d) Find the production rate, in units/hr, when (i) $t = 1$,
        (ii) $t = 2$,
        (iii) $t = 3$.

14. A small crack in a water pipe allows water to escape from the pipe. The number of litres of water that has escaped, t minutes after the crack initially appeared is given by V, where $V = \frac{t}{1000}(t + 10)$.

(a) What volume of water leaked in (i) the first ten minutes,
(ii) the first twenty four hours?

(b) At what rate, in L/min, is the water leaking after (i) 10 minutes,
(ii) 2 hours,
(iii) 24 hours?

15. A wildlife park is involved in a captive breeding program for an endangered species of deer. The program plans to release deer from the herd back into the wild as well as increasing the captive herd's population. The park starts with a captive population of forty deer and the breeding and release back into the wild will be such that the captive population P, t years later, approximately follows the mathematical rule:

$$P = 40 + \frac{t(t + 20)}{10}$$

(a) What will be the captive population after (i) 1 year,
(ii) 2 years,
(iii) 3 years,
(iv) 10 years?

(b) Find an expression for the rate of change of P with respect to t.

(c) Find the rate of change of P with respect to t (in deer/year) after
(i) 5 years,
(ii) 10 years,
(iii) 20 years.

16. Following the survey of a particular mine, experts predict that with continued mining the quantity (T tonnes) of a particular ore remaining in the mine, t years after the survey was carried out would approximately fit the mathematical model

$$T = 20t^3 - 420t^2 - 8\,000t + 150\,000$$

(a) What quantity of ore was in the mine when $t = 0$?

(b) What quantity of ore will be in the mine when $t = 10$?

(c) Find a rule for the rate of decrease of T in tonnes per year.

(d) Calculate the rate that T will be decreasing in tonnes per year when
(i) $t = 2$,
(ii) $t = 4$,
(iii) $t = 7$.

17. An automatic puncture repair compound for bicycle tyres is being tested. The idea is that the compound can exist in vaporised form, mixed with the air in the tyre. When a puncture occurs the tyre pressure forces air out through the hole. This flow of air through the hole causes the glue-like compound to condense at the hole and repair the puncture.
The compound is tested in a specially constructed balloon. The balloon is punctured and its volume is noted as it deflates. In the test the balloon had a volume $V\,\text{cm}^3$ where

$$V \approx 1000 - 4t + \frac{1}{10}t^2$$

with t being the time in seconds since the puncture occurred.

Determine (a) the volume of the balloon when the puncture occurred,
(b) the volume of the balloon two seconds after the puncture occurred,
(c) an expression for the rate of change of V with respect to t,
(d) the instantaneous rate of change of volume, in cm^3/sec, when
(i) the puncture occurs,
(ii) $t = 3$,
(e) how long it takes the compound to repair the puncture.

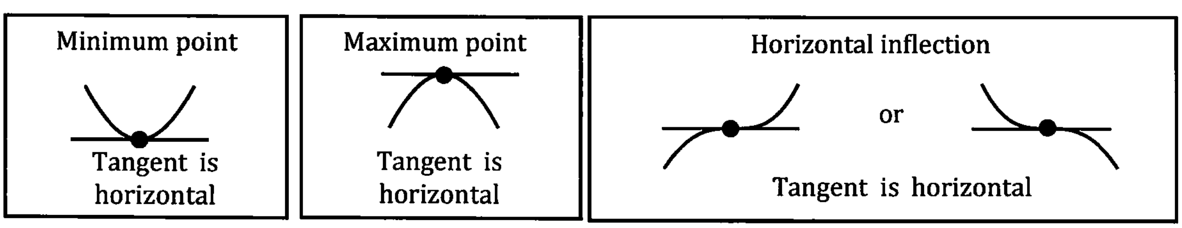
(f) The given formula for V can only make sense for $a \le t \le b$. Find a and b.

Using differentiation to locate stationary points of polynomial functions.

Note carefully the following points as the ideas are used in the examples that follow to locate the stationary points on the graphs of polynomial functions and to determine the nature of such points.

- At local maximum points local minimum points and at points of horizontal inflection the gradient of the curve is momentarily zero.

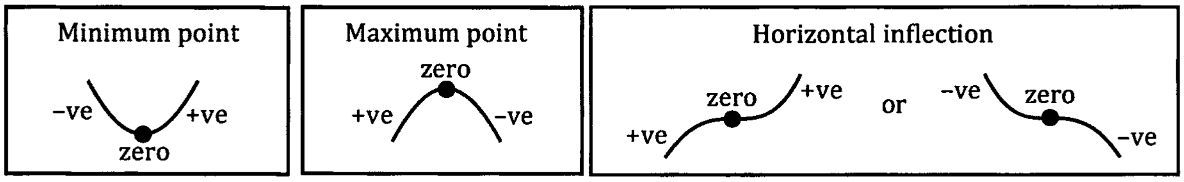
  I.e. $\dfrac{dy}{dx}$ is zero at these points.



- As we pass through a minimum turning point (in the direction of increasing x) the gradient changes from negative to positive.

  As we pass through a maximum turning point (in the direction of increasing x) the gradient changes from positive to negative.

  For horizontal inflection the gradient, though momentarily zero, does not change sign.



Example 4

For the function $y = x^2 + 6x - 4$ use differentiation to determine the nature and location of any stationary points.

If $\quad y = x^2 + 6x - 4$

then $\quad \frac{dy}{dx} = 2x + 6$

At stationary points $\quad \frac{dy}{dx} = 0 \quad \therefore \quad 2x + 6 = 0$

i.e. $\quad x = -3$

When $x = -3$ $\quad y = (-3)^2 + 6(-3) - 4$

$= 9 - 18 - 4$

$= -13$

There is a stationary point at (–3, –13).

Consider gradient either side of $x = -3$:

	$x = -3{\cdot}1$	$x = -3$	$x = -2{\cdot}9$
$2x + 6$	–ve	zero	+ve
	\	—	/

Thus $y = x^2 + 6x - 4$ has a minimum turning point at (–3, –13).

Alternatively we could use our familiarity with the graphs of quadratic functions to state that the stationary point is a minimum because the coefficient of x^2 in the quadratic function is positive.

Example 5

For the function $y = 2x^3 - 6x^2$, and without the use of a calculator, determine

(a) the coordinates of any points where the graph of the function cuts the y-axis,

(b) the coordinates of any points where the graph of the function cuts (or touches) the x-axis,

(c) the behaviour of the function as $x \to \pm\infty$.

(d) the nature and location of any stationary points on the graph of the function.

Hence sketch the graph of the function.

(a) On the y-axis, $x = 0$

If $x = 0$
$$y = 2(0)^3 - 6(0)^2 = 0$$

The graph of the function cuts the y-axis at (0, 0).

(b) On the x-axis, $y = 0$

If $y = 0$
$$2x^3 - 6x^2 = 0$$
i.e.
$$2x^2(x-3) = 0$$
$$x = 0 \text{ or } 3$$

The graph of the function cuts (or touches) the x-axis at (0, 0) and (3, 0).

(c) As x "gets large" the x^3 term will dominate.

Thus as $x \to \infty$ $y \to \infty$ (and faster than x does).

and as $x \to -\infty$ $y \to -\infty$ (and faster than x does).

(d) If
$$y = 2x^3 - 6x^2$$
then
$$\frac{dy}{dx} = 6x^2 - 12x = 6x(x-2)$$

At stationary points $\frac{dy}{dx} = 0$ $\therefore$ $6x(x-2) = 0$

i.e. $x = 0$ or 2

When $x = 0$, $y = 2(0)^3 - 6(0)^2 = 0$ and when $x = 2$, $y = 2(2)^3 - 6(2)^2 = -8$

There are stationary points at (0, 0) and at (2, −8).

Consider gradient either side of $x = 0$.

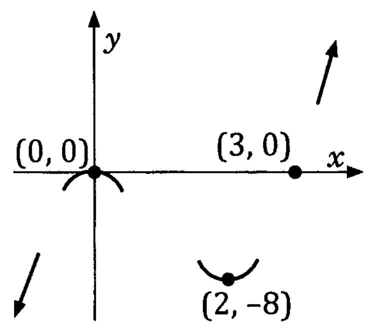
	$x = -0{\cdot}1$	$x = 0$	$x = 0{\cdot}1$
$6x(x-2)$	+ve	zero	−ve
	/	—	\

Consider gradient either side of $x = 2$

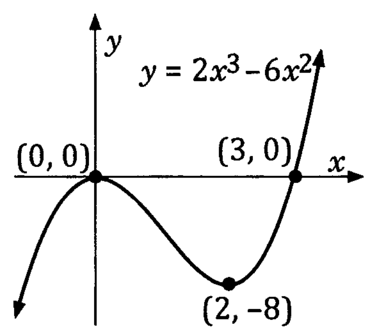
	$x = 1{\cdot}9$	$x = 2$	$x = 2{\cdot}1$
$6x(x-2)$	−ve	zero	+ve
	\	—	/

Thus $y = 2x^3 - 6x^2$ has a maximum turning point at (0, 0) and a minimum turning point at (2, −8).

(e) The information from the previous parts of this question can be placed on a graph as shown below.

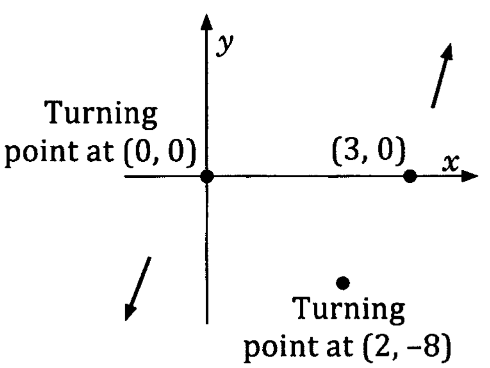


A sketch can be completed:



The reader should check the reasonableness of this sketch by viewing the graph of this function on a graphic calculator.

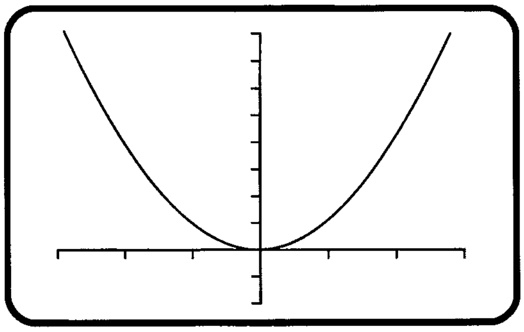
Note • In the previous example the nature of each stationary point was determined by examining the sign of the gradient. We could have determined the nature of each point just from the shape that the sketch had to be to satisfy the behaviour as $x \to \pm\infty$ and the location of the turning points. From the diagram on the right (0, 0) must be a local maximum and (2, -8) must be a local minimum.



- Specific facilities on graphic calculators allow stationary points of polynomials, and other functions, to be readily located without the need to use differentiation. This ability is very useful but also needs care. In some cases the portion of the graph the display is showing may not be telling the whole story.

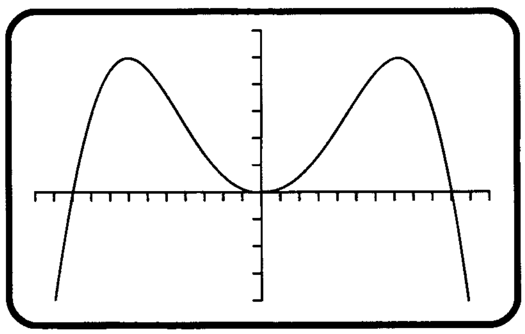
  For example the display on the right is for

  $$y = x^2 - 0{\cdot}01x^4$$

  At first glance it may appear to be similar to $y = x^2$ with a single turning point at (0, 0).

  However, as x gets large we would expect the $-0{\cdot}01x^4$ term to dominate and would thus expect that as $x \to \pm\infty$, $y \to -\infty$.

  Sure enough, zooming out on the calculator we do indeed see that the original picture was not telling the whole story. The graph has three turning points, 2 maximums and one minimum. Using a calculus approach informs us where all of the stationary points are.

- If a question specifically requires that you do not use a calculator, or specifically requires you to show the use of differentiation (or **calculus**, of which differentiation is a part) and algebraic processes then proceed as follows:

  1. Differentiate y with respect to x to obtain $\frac{dy}{dx}$.
  2. Find the values of x for which $\frac{dy}{dx} = 0$.
  3. Find the values of y corresponding to each value of x from 2.
  4. Either by considering the necessary shape of the graph, or by considering the sign of the gradient, determine the nature of the stationary points.

Global maximum and minimum values.

In some cases we may be concerned with the maximum or minimum value a function can take for some interval $a \le x \le b$. We are then concerned with the global maximum (or minimum), which may or may not coincide with the local maximum (or minimum).

Example 6

Using calculus and algebra determine the coordinates and nature of any stationary points on the graph of

$$f(x) = 9x^2 - x^3 - 15x + 11.$$

Hence determine the maximum value of f(x) for (a) $0 \le x \le 7$,
(b) $-2 \le x \le 7$.

If $f(x) = 9x^2 - x^3 - 15x + 11$

then $f'(x) = 18x - 3x^2 - 15$

$= -3(x^2 - 6x + 5)$

$= -3(x-1)(x-5)$

At stationary points $f'(x) = 0$, i.e. $-3(x-1)(x-5) = 0$

giving $x = 1$ or $x = 5$.

If $x = 1$, $f(x) = 4$. If $x = 5$, $f(x) = 36$.

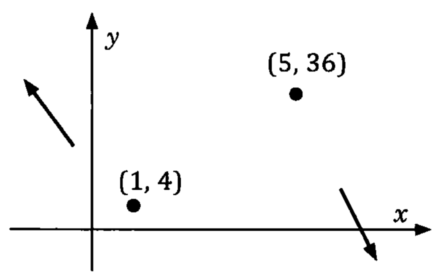
Stationary points occur at (1, 4) and at (5, 36).

With $f(x) = 9x^2 - x^3 - 15x + 11$ the x^3 term will dominate for "large x".

Thus as $x \to +\infty$ $f(x) \to -\infty$

and as $x \to -\infty$ $f(x) \to +\infty$

Thus (1, 4) is a local minimum point and (5, 36) is a local maximum point.



The global maximum will either occur at the local maximum or at the "left end" of the function if the values of x allow us to go sufficiently far to the left.

(a) For $0 \le x \le 7$. $f(0) = 9(0)^2 - (0)^3 - 15(0) + 11$

$= 11$ which does not exceed f(5).

Thus the global maximum for $0 \le x \le 7$ is 36.

(b) For $-2 \le x \le 7$. $f(-2) = 9(-2)^2 - (-2)^3 - 15(-2) + 11$

$= 36 + 8 + 30 + 11$

$= 85$ which does exceed f(5).

Thus the global maximum for $-2 \le x \le 7$ is 85.

Had we not been required to use calculus and algebra the global maximums determined in the previous example could have been obtained using a calculator that is able to determine the maximum value of a function for a given domain.

Again explore the capability of your calculator in this regard.

```
fMax(9·x²-x³-15·x+11,x,0,7)
                    {MaxValue=36, x=5}
fMax(9·x²-x³-15·x+11,x,-2,7)
                    {MaxValue=85, x=-2}
```

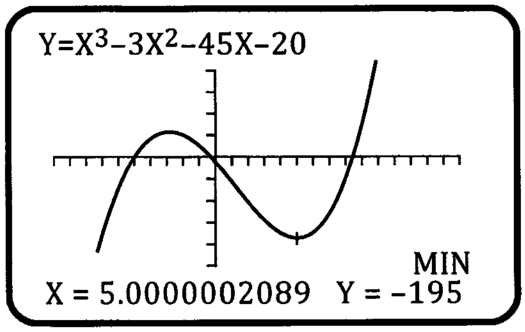
Exercise 6B

1. A student used his old graphic calculator to locate the turning points on the curve

   $$y = x^3 - 3x^2 - 45x - 20.$$

   The display, see right, gave him the coordinates of the local minimum as

   (5·0000002089, −195).

   Use calculus to

   (a) justify that the exact location of this minimum point is (5, −195),

   (b) justify that the turning points displayed are the only ones the curve has,

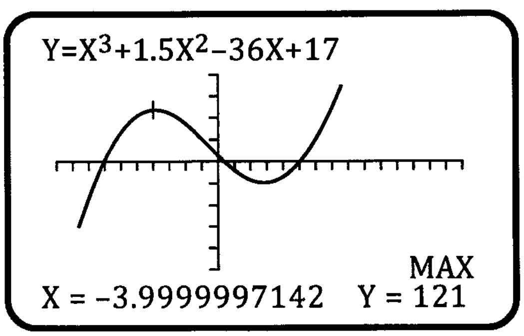
   (c) determine the exact coordinates of the local maximum point.

2. A student used her old graphic calculator to locate the turning points on the curve

   $$y = x^3 + 1{\cdot}5x^2 - 36x + 17.$$

   The display, see right, gave her the coordinates of the local maximum as

   (−3·9999997142, 121).

   Use calculus to

   (a) justify that the exact location of this maximum point is (−4, 121),

   (b) justify that the turning points displayed are the only ones the curve has,

   (c) determine the exact coordinates of the local minimum point.

For questions 3 to 10, and without the aid of a calculator:

- use calculus to determine the coordinates of any stationary points on the graph of the given function,
- indicate the nature of each stationary point,
- produce a sketch of the graph of the function including on your sketch the location of any stationary points, any points where the graph cuts the vertical axis and indicate the behaviour of the graph as $x \to \pm\infty$.

3. $y = x^3 + 3x^2 - 9x - 7$
4. $y = x^3 - 9x^2 + 15x + 30$
5. $y = 1 + 8x - 2x^2$
6. $y = x^5$
7. $y = x^4$
8. $y = 3x^2 - x^3$
9. $y = 2x^2 - 4x + 7$
10. $y = 3x^4 + 4x^3 - 12x^2 + 10$

11. For the function $y = x^3 + 6x^2 + 9x$, and without the use of a calculator, determine
    (a) the coordinates of any points where the graph of the function cuts the y-axis,
    (b) the coordinates of any points where the graph of the function cuts (or touches) the x-axis,
    (c) the behaviour of the function as $x \to \pm\infty$,
    (d) the nature and location of any stationary points on the graph of the function.
    (e) Hence sketch the graph of the function.
    (f) Determine the minimum and maximum value of y for $-5 \le x \le 1$.

12. Use calculus techniques to find the coordinates of any stationary points on the graph of $\quad f(x) = 2x^3 - 3x^2$
    and determine the nature of each.
    Determine the minimum value of $f(x)$ for (a) $x \ge 0$,
    (b) $-1 \le x \le 5$.

Applications.

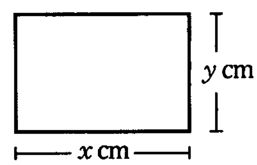
There are many occasions in real life when we look for the most desirable, most favourable or **optimum** situation. Finding the situation that involves maximum profit, most benefit, greatest growth, maximum effect, greatest comfort, greatest speed etc are all situations where the optimum situation involves a maximum. In other situations we might look for the least cost, the least effort, the lowest inflation, the minimum pain, the least discomfort, the slowest speed etc and in such cases are looking for a minimum level. Our ability to use calculus to determine the maximum or minimum values gives us a way of finding the optimum situations in various contexts.

Example 7

What should be the dimensions of a rectangular shape of perimeter 20 cm if its area is to be a maximum?

We require maximum area. If the area is A cm^2 we need a formula A = ???.

Let the required rectangle have dimensions x cm by y cm.



Then $A = xy$. ← ①

Now we cannot differentiate A because the right hand side of equation ① involves two variables x and y.

However we do know that $2x + 2y = 20$

∴ $y = 10 - x$. ← ②

Substituting ② into ① gives $A = 10x - x^2$. ← A quadratic function.

Thus $\dfrac{dA}{dx} = 10 - 2x$

If $\dfrac{dA}{dx} = 0$ then $0 = 10 - 2x$ i.e. $x = 5$

From our knowledge of quadratic functions with a negative coefficient of x^2, or by examining the gradient of the function either side of $x = 5$, we know that $x = 5$ will give a local (and global) maximum.

Thus $x = 5$ gives a maximum value for A.

If $x = 5$ then, from equation ②, $y = 5$.

Thus for maximum area the rectangle should be a square of side 5 cm.

Points to note for solving optimisation problems:

- If a diagram is not given then draw one if it helps.
- Identify the variable that is to be maximised, or minimised. If this variable is, say, C then you must find an equation with C as the subject. i.e. C = ???.
- If this equation for C involves two variables (other than C) find another equation that will allow you to substitute for one of the variables.

- When you have C in terms of one variable, say x, then find the values of x for which

$$\frac{dC}{dx} = 0.$$

- Use your knowledge of what the function must look like (perhaps it is a quadratic and therefore has one turning point ... perhaps the behaviour as x "gets large" helps ... perhaps examine the gradient either side of the stationary point) to determine whether maximum or minimum.

- Check that the value of x for the required maximum, or minimum, is within the values that the situation allows x to lie and check that it gives the global maximum, or minimum.

Example 8

The profit, \$P, made by a company producing and marketing x items of a certain product is given by: $P = -x^3 + 30x^2 + 900x - 1000.$

Clearly showing the use of calculus, find the value of x for maximum profit and determine this maximum profit.

We wish to maximise profit and we have a formula for P in terms of one variable, x. Thus we may differentiate.

$$\begin{aligned} \frac{dP}{dx} &= -3x^2 + 60x + 900 \\ &= -3(x^2 - 20x - 300) \\ &= -3(x - 30)(x + 10) \end{aligned}$$

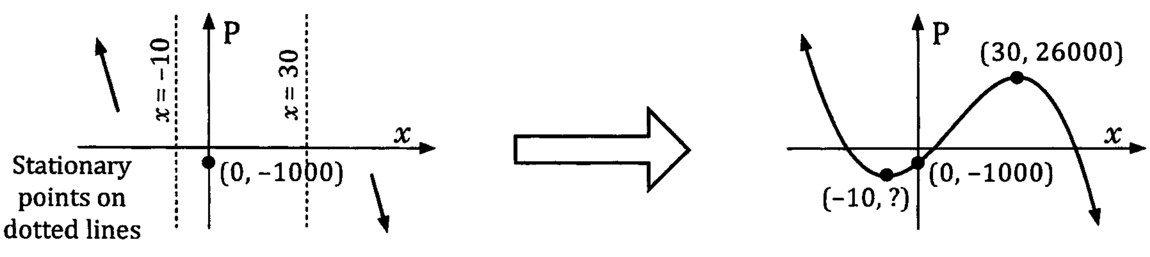
If $\frac{dP}{dx} = 0$, then $(x - 30)(x + 10) = 0$

Solving gives $x = 30$ or -10 (−10 not applicable in this situation)

When $x = 30$, $P = -(30)^3 + 30(30)^2 + 900(30) - 1000$

$= 26\,000.$

If we also consider the y-axis intercept and $x \to \pm\infty$, a sketch can be made:



The sketch indicates that $x = 30$ will give the local *maximum* and for $x \geq 0$ this maximum will not be exceeded elsewhere.

For maximum profit the value of x should be 30 and the maximum profit would then be \$26 000.

Note: In the previous example

- ☞ The nature of the stationary point at $x = 30$ can also be found by considering the gradient either side of $x = 30$:

	$x = 29$	$x = 30$	$x = 31$	
$-3(x-30)(x+10)$	+ve	zero	−ve	
	/	—	\	i.e. maximum point.

- ☞ Viewing the graph of $y = -x^3 + 30x^2 + 900x - 1000$ on a calculator confirms the correctness of the sketch.

Exercise 6C

Use calculus to solve the following optimisation problems. (Use a calculator to assist with the arithmetic if you wish but clearly show the use of calculus to locate the optimum situation.)

1. If $X = t^3 - 15t^2 + 48t + 80$ find the value of t for which X has a local minimum value and find this minimum.

2. If $A = 60p + 12p^2 - p^3 - 500$ find the value of p for which A has a local maximum value and find this maximum.

3. If $A = xy$ and $x + 5y = 20$ find the maximum value of A and the values of x and y for which this maximum value occurs.

4. If $A = xy$ and $2x + 3y = 18$ find the maximum value of A and the values of x and y for which this maximum value occurs.

5. The total cost, \$$C$, and total revenue, \$$R$, arising from the production and marketing of x items of a certain product are given by
   $$R = x(95 - x) \qquad \text{and} \qquad C = 500 + 25x.$$
   Given that: Profit = Revenue − Cost find the value of x that gives maximum profit and determine what this maximum profit will be.

6. The total cost for the production and marketing of x items of a certain product is \$$C$ where $C = 5000 + 60x$.
   The revenue received from each item is \$$R$ where $R = 300 - x$.
   Given that: Profit = Revenue − Cost find the value of x that gives maximum profit and determine what this maximum profit will be.

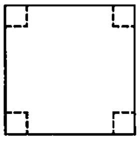
7. The organisers of a sheepdog competition have 100 metres of fencing available to fence an enclosure for some sheep. They wish to make the area rectangular and as large as possible. What dimensions should the enclosure have to maximise area if
   (a) the 100 m of fencing is to be used for all four sides,
   (b) an existing wall forms one side and the fencing is used for the other three.

8. A manufacturer wishes to advertise a new product. He knows that advertising will increase sales but the advertising itself costs money. From past experience with a similar product the manufacturer expects that his profit \$P, after the advertising has been paid for, will be related to x, the number of thousands of dollars spent on advertising according to the rule $P = 50000 + 6000x - 100x^2$.
How much should the manufacturer spend on advertising in order to maximise profit and what would this maximum profit be?

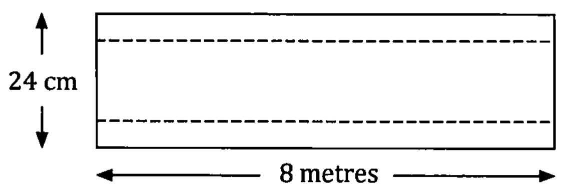
9. A rectangular box is to be made to the following requirements:
 - The length must be one and a half times the width.
 - The twelve edges must have a total length of 6 m.

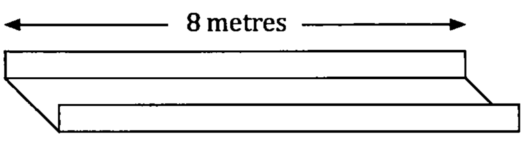
 Find the dimensions of the box that meets these requirements and that maximise the capacity.

10. An open cardboard box is to be made by cutting squares of side x cm from each corner of a square of card of side 60 cm and folding the resulting "flaps" up to form the box. Find the value of x that will give the box a maximum capacity.



11. A long narrow sheet of metal, 8 metres by 24 cm, is to be made into a gutter by folding up equal widths of metal along each edge of the sheet to form the two identical vertical walls (see diagram).





Use differentiation to determine how many centimetres should be turned up along each edge to maximise the capacity of the gutter for carrying water?

12. The organisers of a raffle are trying to decide the price they should charge for tickets. From past experience they feel confident that they can sell 7500 tickets if they charge \$1 per ticket. For each 10 cent rise in the price they estimate that they will sell 250 tickets less.
They need to raise \$5000 to cover the cost of prizes and printing.
If they set the price per ticket at $\$(1 + 0{\cdot}1x)$, i.e. \$1 plus x lots of 10 cents, find
(a) an expression in terms of x for the profit the raffle will raise,
(b) the value of x for maximum profit.
For this maximum profit situation find
(c) the price of each ticket,
(d) the number of tickets they can expect to sell,
(e) the maximum profit.

13. A colony of bacteria is monitored in a laboratory over a 24 hour period ($0 \le t \le 24$) and its population, N, at time t was found to approximately follow the rule

$$N = 2t^3 - 57t^2 + 288t + 2900.$$

Determine the minimum and maximum value of N (nearest 100) for $0 \le t \le 24$.

14. A body is projected from an origin O and moves in a straight line such that its distance from O, t seconds after projection, is s metres where

$$s = \frac{t^3}{3} - 6t^2 + 50t, \quad (t \ge 0).$$

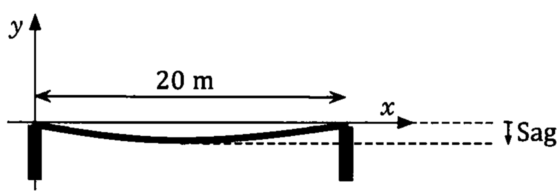
The velocity of the body, v m/sec, t seconds after projection is given by $\frac{ds}{dt}$.

(a) How far is the body from the origin after three seconds?
(b) Find an expression for the velocity of the body t seconds after projection.
(c) What is the initial (i.e. $t = 0$) velocity of the body?
(d) For what value of t ($t \ge 0$) is the body moving with minimum velocity and how far from O is the body at this time?

15. One small part of a construction project involves a metal rod spanning a 20 metre gap with the rod resting on supports at each end. The rod, which is not uniform, is expected to "sag" somewhat under its own weight. The mathematical model of the situation predicts that this rod will take the shape of the curve

$$y = \frac{x}{50000}(20 - x)(x - 50), \quad \text{for } 0 \le x \le 20,$$

with x and y axes as shown in the diagram.



Clearly showing the use of calculus, but using your calculator to solve any equations that may result, determine the maximum sag in the rod (to the nearest mm) and where it occurs (as a distance from the origin, to the nearest centimetre).

16. The owner of a large house decided to spend some money making it more secure. An expert analysed the situation and said that for \$5000 the security rating, R, of the property would rise from its current score of 30 points to 100 points. Every \$500 spent after that would lift the rating by 5 points.

The owner feels that the more security devices he pays for the more tedious he is going to find it to enter and leave his property with all that he will have to remember to lock/unlock, arm/disarm etc. He feels there is an owner convenience rating, C, which will go down by 2 points, from an initial 100 points, for every \$500 he spends over the \$5000 that he accepts is necessary.

As both the security rating and the convenience rating interest him he decides to multiply them together to form the "secure but not inconvenient" rating Z.

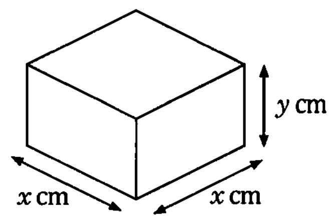
Clearly showing your use of calculus, determine how much the owner should spend on security to maximise Z.

Applications - extension.

The next example, and the exercise that follows involve applications of differentiation for which the function involved contains negative or fractional powers. At the time of writing this text the syllabus for this unit includes optimisation problems for polynomial functions only. Hence the optimisation questions in this next section should be regarded as an extension activity.

Example 9

The rectangular block shown on the right has a square base of side x cm, a height of y cm and a volume of 80 cm^3. The base and top are to be covered with lacquer costing 5 cents/cm^2 and the sides with lacquer costing 4 cents/cm^2. Find the values of x and y for minimum cost.



To minimise cost we need a formula Cost = ???.

Cost of lacquering the base = $5x^2$ cents.

Cost of lacquering each side = $4xy$ cents.

Thus if the total cost is C cents then $C = 10x^2 + 16xy$ ← 1

We cannot differentiate C at present because it involves two variables, x and y.

However we know that $x^2y = 80$ (because the volume = 80 cm^3).

i.e. $y = \frac{80}{x^2}$ ← 2

Substituting from 2 into 1 gives $C = 10x^2 + \frac{1280}{x}$

Thus $\frac{dC}{dx} = 20x - \frac{1280}{x^2}$

At stationary points $\frac{dC}{dx} = 0$, $\therefore$ $0 = 20x - \frac{1280}{x^2}$ giving $x = 4$.

From 2, if $x = 4$, $y = 5$ (and C = 480).

Given the context of the question x cannot be negative.

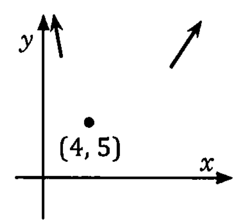
Considering $x \geq 0$:

With $C = 10x^2 + \frac{1280}{x}$ the y-axis will be an asymptote.

As $x \to 0$ (from the positive side), $y \to +\infty$.

As $x \to \infty, y \to +\infty$.

Thus (4, 5) must be a global minimum.



(Alternatively consider the sign of the gradient either side of $x = 4$.)

Thus to minimise cost the block should be made with a base 4 cm by 4 cm and a height of 5 cm, (i.e. $x = 4$ and $y = 5$).

Exercise 6D

1. If $P = 3r^2 + \frac{5}{r}$ find an expression for the rate of change of P with respect to r.

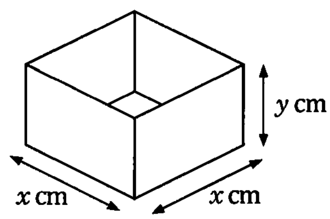
2. If $A = 400\sqrt{r}$ find the rate of change of A with respect to r when
   (a) $r = 4$, (b) $r = 25$, (c) $r = 100$.

3. Use calculus to locate exactly the stationary points on the graph of
$$y = x + \frac{2}{x}$$
and determine the nature of each by consulting a graphic calculator display.

4. Use calculus techniques to determine the exact coordinates of any stationary points on the curve
$$y = 5 - \frac{4}{x} - x$$
and, by considering the sign of the gradient on either side of any turning points determine whether maximum, minimum or horizontal inflection.

5. Use calculus techniques to determine the exact coordinates of any stationary points on the curve $y = 3x - \frac{96}{x^2}$.
By considering the graph of the function for x close to zero and $x \to \pm\infty$ determine whether maximum, minimum or horizontal inflection.

6. The open rectangular box shown on the right is to have a square base of side x cm and a height y cm. The volume of the box is to be 500 cm^3.



   (a) Find an expression for y in terms of x.
   (b) The box is to be made of card. Find, in terms of x, the area of card required to make each box, assuming no wastage.
   (c) Find x and y for which this area is a minimum and then find this minimum area.

7. A food manufacturer wishes to package a product in cylindrical tins each of volume 535 cm^3. Find the base radius and height of tins that meet this volume requirement and that minimise the metal required to make them, i.e. minimum surface area. Give your answers in centimetres and correct to 1 decimal place.

8. A metal box company is asked to produce cylindrical metal tins, each with a volume of 535 cm^3. The base and top of each tin have to be made from thicker material than is used for the wall. This thicker material costs twice as much per cm^2 as the thinner material. Find, in centimetres and correct to one decimal place, the base radius and height of each tin for the cost of material to be a minimum.

Miscellaneous Exercise Six.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

1. Express each of the following as a power of 5.

(a) 25	(b) 625	(c) 125
(d) 1	(e) $5 \times 5 \times 5$	(f) $5 \times 5 \times 5 \times 5 \times 5 \times 5$
(g) $(5 \times 5) \times (5 \times 5 \times 5)$	(h) $(5 \times 5 \times 5) \times 5$	(i) $(5 \times 5 \times 5)^2 \times 5$
(j) $\dfrac{5 \times 5 \times 5 \times 5 \times 5}{5 \times 5}$	(k) $\dfrac{5 \times 5 \times 5}{5}$	(l) $\dfrac{(5 \times 5 \times 5) \times (5 \times 5)}{5 \times 5 \times 5 \times 5}$
(m) $5^3 \times 5^7$	(n) $5^3 \times 5$	(o) $5^3 \times 5^7 \times 5^7$
(p) $5^5 \div 5^3$	(q) $5^8 \div 5^2$	(r) $5^{11} \div 5^8$
(s) $5^4 \times 5^3 \div 5^2$	(t) $5^3 \times 5^4 \div 5$	(u) $5^8 \div 5^3 \times 5^2$
(v) $5^5 \times 125$	(w) $5^5 \div 125$	(x) $5^8 \div (5^3 \times 5^2)$
(y) $3^2 + 4^2$	(z) $\dfrac{6^2 + 7 \times 2}{3 - 1}$	

2. Without the assistance of a calculator, simplify each of the following, expressing you answers in terms of positive indices.

(a) $\dfrac{(a^3 \times \sqrt{a})^2}{a^3}$	(b) $\dfrac{(5b^{-2}a)^3}{25a^{-4}b^2}$	(c) $\dfrac{2^{n+1} + 2^{2n}}{2^n}$
(d) $\dfrac{5x^4 + 10x^7}{5x^3}$	(e) $\dfrac{2^x + 2^{x+3}}{9}$	(f) $\dfrac{3^{n+1} - 15}{5 \times 3^n - 25}$

3. An arithmetic progression is such that
   - the fourth term is 130

   and
   - $T_{n+1} = T_n + 11$.

   Determine the first six terms of the sequence.

4. A geometric progression is such that
   - the fourth term is 2·8

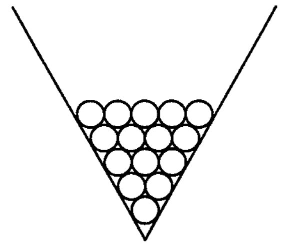
   and
   - $T_{n+1} = T_n \times 0{\cdot}2$.

   Determine the first five terms of the sequence.

5. Find
   (a) the average rate of change of the function $y = x^2 - 3x$ from the point P(3, 0) to the point Q (6, 18),
   (b) the instantaneous rate of change of $y = x^2 - 3x$ at the point P (3, 0),
   (c) the instantaneous rate of change of $y = x^2 - 3x$ at the point Q (6, 18).

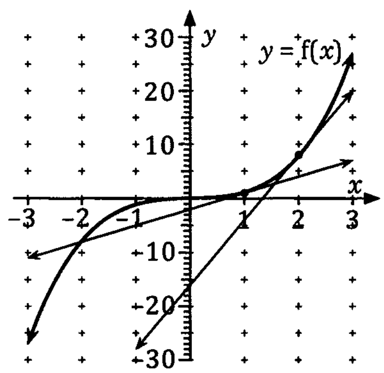
6. Find (a) the average rate of change of the function $y = x^3 - 3x$ from the point P(3, 18) to the point Q (6, 198),
   (b) the instantaneous rate of change of $y = x^3 - 3x$ at the point P (3, 18),
   (c) the instantaneous rate of change of $y = x^3 - 3x$ at the point Q (6, 198).

7. The diagram on the right shows a device for counting pills.
   The diagram shows the device containing 5 rows of pills.
   How many pills are shown in the device?
   How many pills would be in such a device if it were to contain (a) 10 complete rows,
   (b) 15 complete rows.



8. Evaluate the following sums:
   (a) $3 + 12 + 21 + 30 + 39 + 48 + \ldots\ldots + 507$
   (b) $S_{10} = T_1 + T_2 + T_3 + T_4 + \ldots + T_{10}$
   $= 30 - 90 + 270 - 810 + \ldots - 590490$
   (c) $6 + 12 + 24 + 48 + 96 + \ldots\ldots + 6291456$
   (d) $100 + 80 + 64 + 51{\cdot}2 + 40{\cdot}96 + 32{\cdot}768 + \ldots\ldots$
   (e) $5 - 5 + 5 - 5 + 5 - 5 + 5 - 5 + 5 \ldots\ldots + 5$

9. The graph on the right shows a curve $y = f(x)$ with the tangents at $x = 1$ and at $x = 2$ drawn.
   Use the tangents to determine the gradient of $y = f(x)$ at $x = 1$ and at $x = 2$.
   As you may have realised, the graph is that of $y = x^3$.
   Check your previous answers using calculus.



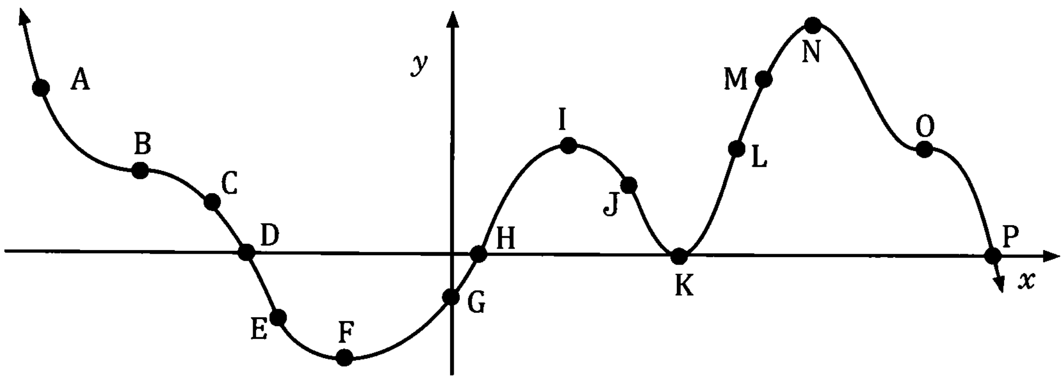
10. A curve is such that $\frac{dy}{dx} = (x + 4)(2x - 3)$.
    At how many places on the curve is the gradient zero?

11. Use the formula:

$$\text{Gradient at P}\ (x, f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

to determine
   (a) the instantaneous rate of change of the function $f(x) = x^2$ when $x = 5$.
   (b) the instantaneous rate of change of the function $f(x) = x^2 + x$ when $x = 1$.
   (c) the instantaneous rate of change of the function $f(x) = x^3 + x$ when $x = 2$.

12. For the graph shown below state which of the points marked A → P are
   (a) places on the curve where the function is zero, (4 points),
   (b) places on the curve where the gradient is zero, (6 points),
   (c) places on the curve where the gradient is positive, (4 points),
   (d) places on the curve where the gradient is negative, (6 points).



13. Where on the curve $y = x^2 + 5x - 4$ is the gradient the same as the gradient of the line with equation $y = 7x - 3$?

14. Find the coordinates of the point(s) on the following curves where the gradient is as stated. (a) $y = 2x + \frac{1}{x}$. Gradient 1. (b) $y = 3x - 4\sqrt{x}$. Gradient −1.

15. For $f(x) = 2x^4 - 5x^3 + x^2 - 2x + 6$ use your calculator to determine
   (a) f(21), f(31) and f(41). (b) f'(21), f'(31) and f'(41).

16. Clearly showing your use of differentiation and algebra find the equations of the tangents to the curve

$$y = x^3 + 3x^2 - 20x + 10$$

at any points on the curve where the gradient is equal to 25.

17. A butcher normally sells chicken fillets for \$10·50 per kg. During a week in which she has them on special for \$9·50 per kg she finds that her usual sales of 50 kg per week jumps to 70 kg per week.
   (a) Assuming that the number of kg sold per week, *N*, obeys a rule of the form

$$N = ap + c$$

   where \$*p* is the price per kg and a and c are constants, find a and c.
   (b) Write an expression in terms of *p* for the total revenue the butcher receives for selling *N* kg at \$*p* per kg.
   (c) If the butcher pays \$7 per kg for the fillets write down an expression in terms of *p* for the profit she makes from buying and selling *N* kg.
   (d) Find the value of *p* for maximum profit and, for this value of *p*, determine the number of kg sold and the profit.

18. Given that $p = 7 \times 10^{12}$ and $q = 2 \times 10^{11}$ evaluate each of the following, without the assistance of a calculator, giving your answers in standard form (scientific notation).

(a)	$p \times q$	(b)	$p + q$	(c)	$p - q$
(d)	$p \div q$	(e)	$5pq$	(f)	$\frac{p^2}{q}$

19. N, the number of organisms present in a certain culture of bacteria, t hours after observation commenced was found to approximately follow the rule

$$N = t^3 + 30t + 200.$$

Find
(a) the value of N when observation commenced,
(b) the value of N when $t = 10$,
(c) the average number of new organisms produced per hour during the first ten hours of observation,
(d) the instantaneous rate of change of N (in organisms per hour) when
(i) $t = 0$, (ii) $t = 5$, (iii) $t = 10$.

20. Explain what each of the following displays tell us about the rate of change of

$$f(x) = x^4 + x$$

(a)

Define $f(x) = x^4 + x$

$\frac{f(3) - f(1)}{2}$

41

(b)

Define $f(x) = x^4 + x$

$\lim_{h \to 0} \left(\frac{f(3 + h) - f(3)}{h} \right)$

109

21. The manufacturer of a certain fabric estimates that she can sell 500 m of the fabric each week if the price is \$10 per metre. However, market research indicates that each 20 cents per metre price reduction will increase sales by 25 metres. If the manufacturer reduces the cost per metre by x lots of 20 cents find
(a) an expression for the cost per metre,
(b) an expression for the number of metres sold,
(c) an expression for the total revenue (income),
(d) the value of x that makes this total revenue a maximum and explain how you know that your value of x will give a *maximum*.